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Property (w) for perturbations of polaroid operators[☆]

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Abstract

A bounded linear operator $T \in L(X)$ acting on a Banach space satisfies property (w), a variant of Weyl's theorem, if the complement in the approximate-point spectrum $\sigma_a(T)$ of the Weyl essential approximate-point spectrum $\sigma_{wa}(T)$ is the set of all isolated points of the spectrum which are eigenvalues of finite multiplicity. In this note, we study the stability of property (w) for a polaroid operator T acting on a Banach space, under perturbations by finite rank operators, by nilpotent operators and, more generally, by algebraic operators commuting with T.
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1. Definitions and basic results

Let X denote an infinite-dimensional complex Banach space, and denote by $L(X)$ the algebra of all bounded linear operators on X. If $T \in L(X)$ by $\alpha(T)$ we shall denote the dimension of the kernel $\ker T$ and by $\beta(T)$ the codimension of the range $T(X)$. Recall that $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $T(X)$ is closed and $\alpha(T) < \infty$, while $T \in L(X)$ is called

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lower semi-Fredholm, $T \in \Phi_-(X)$, if $\beta(T) < \infty$. The class of all semi-Fredholm operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, while the class of all Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. If $T \in \Phi_\pm(X)$, the *index* of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. Define

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leqslant 0\},$$

and

$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geqslant 0\}.$$

The set of *Weyl operators* is defined by

$$W(X) := W_+(X) \cap W_-(X) = \{T \in \Phi(X) : \operatorname{ind} T = 0\}.$$

The classes of operators defined above generate the following spectra. The *Weyl spectrum* is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\},$$

while the *Weyl essential approximate-point spectrum* is defined by

$$\sigma_{\mathrm{wa}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\},$$

and analogously, the *Weyl essential surjectivity spectrum* is defined by

$$\sigma_{\mathrm{ws}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\},$$

Note that $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{wa}}(T) \cup \sigma_{\mathrm{ws}}(T)$. By duality we also have $\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{ws}}(T^*)$ and $\sigma_{\mathrm{wa}}(T^*) = \sigma_{\mathrm{ws}}(T)$. The classical *approximate-point spectrum* is defined by

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not injective or has not closed range}\},$$

while the *surjective spectrum* is defined by

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\}.$$

It is well-known that $\sigma_{\mathrm{a}}(T^*) = \sigma_{\mathrm{s}}(T)$ and $\sigma_{\mathrm{s}}(T^*) = \sigma_{\mathrm{a}}(T)$. The *ascent* of $T \in L(X)$ is defined as $p := p(T) = \inf\{n \in \mathbf{N} : \ker T^n = \ker T^{n+1}\}$, while the *descent* is defined as $q := q(T) = \inf\{n \in \mathbf{N} : T^n(X) = T^{n+1}(X)\}$, where the infimum over the empty set is taken ∞. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ [14, Proposition 38.3]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T, see [14, Proposition 50.2].

The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T) = q(T) < \infty\}.$$

By Theorem 3.4 of [1] we have $B(X) \subseteq W(X)$. The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}.$$

Clearly, $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ for all $T \in L(X)$, and if $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$ then λ is isolated in $\sigma(T)$. For a bounded operator $T \in L(X)$, define

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and

$$\pi_{00}^a(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Following Coburn [9], we shall say that $T \in L(X)$ satisfies *Weyl's theorem* if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T). \tag{1}$$

The following two interesting variants of Weyl's theorem have been introduced by Rakočević [20,21] and studied in the recent paper [6].

Definition 1.1. A bounded operator $T \in L(X)$ is said to satisfy property (w) if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \pi_{00}(T),$$

while $T \in L(X)$ is said to satisfy a-Weyl's theorem if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \pi_{00}^{a}(T).$$

As observed in [6], we have either of a-Weyl's theorem or property (w) for $T \Rightarrow$ Weyl's theorem holds for T.

Examples of operators satisfying Weyl's theorem but not property (w) may be found in [6]. Property (w) is independent from a-Weyl's theorem: In [6] there are examples of operators $T \in L(X)$ satisfying property (w) but not a-Weyl's theorem and vice versa. Note that property (w) is fulfilled by a relevant number of Hilbert space operators, see [6]. For instance, property (w) is satisfied by generalized scalar operators, and whenever the Hilbert space adjoint T' has property $H(p)$ [6, Corollary 2.20] (the terminology is defined in the next section). Generally, property (w), as well as Weyl's theorems, do not survive under perturbations. More can be said: Weyl's theorems and property (w) for a bounded operator T are liable to fail also under "small" perturbations K, if "small" is interpreted in the sense of compact or quasi-nilpotent operators. In [3] some sufficient conditions are given for which we have the stability of property (w), under perturbations by finite rank operators, compact operators, or quasi-nilpotent operator commuting with T. In this paper we shall prove that the stability of property (w) holds whenever T belongs to some special classes of operators and K is a commuting algebraic operator that commutes with T.

2. Results

Recall that the operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Clearly, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$. Moreover, the identity theorem for analytic functions implies that $T \in L(X)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has SVEP at every isolated point of the spectrum. Note that

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{2}$$

dually

$$q(\lambda I - T) < \infty \Rightarrow T^{*} \text{ has SVEP at } \lambda. \tag{3}$$

Also

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda. \tag{4}$$

Remark 2.1. The implications (2), (3) and (4) are equivalences whenever $\lambda I - T \in \Phi_{\pm}(X)$, see [1, Theorem 3.16].

Theorem 2.2. [6] *Let $T \in L(X)$. Then the following equivalences hold*:

(i) *If T^* has SVEP, then property (w), Weyl's theorem, and a-Weyl's theorem for T are equivalent.*
(ii) *If T has SVEP, then property (w), Weyl's theorem and a-Weyl's theorem for T^* are equivalent.*

Remark 2.3. In the case of Hilbert space operators $T \in L(H)$ instead of the dual T^* it is more appropriate to consider the Hilbert space adjoint T'. It should be noted that T^* has SVEP is equivalent to saying that T' has SVEP, see [2, Section 4].

The following result will be useful in the sequel.

Theorem 2.4. *Suppose that $T \in L(H)$, H a Hilbert space. Then property (w) holds for T^* if and only if property (w) holds for T'.*

Proof. (i) By means of the classical Fréchet–Riesz representation theorem we know that if U is the conjugate-linear isometry that associates to each $y \in H$ the linear form $x \to \langle x, y\rangle$ then

$$\bar{\lambda} I - T' = (\lambda I - T)' = U^{-1}(\lambda I - T)^* U. \tag{5}$$

This obviously implies that $\alpha(\bar{\lambda} I - T') = \alpha(\lambda I^* - T^*)$ for all $\lambda \in \mathbb{C}$. Since $\sigma(T') = \overline{\sigma(T)} = \overline{\sigma(T^*)}$ and analogously $\sigma_{\mathrm{a}}(T') = \overline{\sigma_{\mathrm{a}}(T^*)}$, it then easily follows that the equality $\pi_{00}(T') = \overline{\pi_{00}(T^*)}$ holds.

Suppose now that T^* satisfies property (w). From (5) we also have $\sigma_{\mathrm{aw}}(T') = \overline{\sigma_{\mathrm{aw}}(T^*)}$, so

$$\begin{aligned}\sigma_{\mathrm{a}}(T') \setminus \sigma_{\mathrm{aw}}(T') &= \overline{\sigma_{\mathrm{a}}(T^*)} \setminus \overline{\sigma_{\mathrm{aw}}(T^*)} = \overline{\sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{aw}}(T^*)} \\ &= \overline{\pi_{00}(T^*)} = \pi_{00}(T'),\end{aligned}$$

so T' satisfies property (w). The opposite implication follows in a similar way. $\square$

A bounded operator $T \in L(X)$ is said to be *polaroid* if $\operatorname{iso}\sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of T. Weyl' theorem for polaroid operators has been studied in a very recent paper by Duggal [11].

Theorem 2.5. *If $T \in L(X)$, X a Banach space, is polaroid then its dual T^* is polaroid. If $T \in L(H)$, H a Hilbert space, then T is polaroid if and only if its Hilbert space adjoint T' is polaroid.*

Proof. Suppose that $\lambda \in \operatorname{iso}\sigma(T^*)$. Since $\sigma(T) = \sigma(T^*)$ then λ is an isolated point of $\sigma(T)$ so, by assumption, a pole of the resolvent of T. Let $p := p(\lambda I - T) = q(\lambda I - T)$. By Theorem 3.6 of [1] then

$$X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X). \tag{6}$$

Denote by $M^{\perp}$ the anihilator of a subset M of X. Note that $(\lambda I - T)^p(X)$ is closed, since it is the kernel of the spectral projection associated with the spectral set $\{\lambda\}$, see Theorem 3.74 of

[1], hence by the classical closed range theorem $(\lambda I - T)^p(X)^\perp = \ker(\lambda I^* - T^*)^p$. From the decomposition (6) we then obtain

$$X^* = \ker\,(\lambda I - T)^{p\perp} \oplus (\lambda I - T)^p(X)^\perp = (\lambda I^* - T^*)^p(X^*) \oplus \ker(\lambda I^* - T^*)^p,$$

and hence, again by [1, Theorem 3.6], $p(\lambda I^* - T^*) = q(\lambda I^* - T^*) < \infty$, thus λ is a pole of the resolvent of T^*.

Suppose that $T \in L(H)$ is polaroid. To prove that T' is polaroid let λ be an isolated point in $\sigma(T') = \overline{\sigma(T)}$. Then $\bar{\lambda}$ is isolated in $\sigma(T)$, and hence $\bar{\lambda}$ is a pole of the resolvent of T, thus $p := p(\bar{\lambda} I - T) = q(\bar{\lambda} I - T) < \infty$. Consequently, $H = \ker(\bar{\lambda} I - T)^p \oplus (\bar{\lambda} I - T)^p(X)$ and the range $(\bar{\lambda} I - T)^p(X)$ is closed. From this it then follows that

$$H = (\ker \bar{\lambda} I - T)^{p\perp} \oplus (\bar{\lambda} I - T)^p(H)^\perp = (\lambda I - T')^p(H) \oplus \ker(\lambda I - T')^p,$$

where here $N^\perp$ denotes the orthogonal complement of $N \subseteq H$. Therefore $p(\lambda I - T') = q(\lambda I - T') < \infty$, or equivalently λ is a pole of the resolvent of T', thus T' is polaroid. Conversely, if T' is polaroid by the first part of the proof then $T'' = T$ is polaroid. □

Theorem 2.6. *Suppose that $T \in L(X)$.*

(i) *If T^* has SVEP then $\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{b}}(T)$.*
(ii) *If T has SVEP then $\sigma_{\mathrm{wa}}(T^*) = \sigma_{\mathrm{b}}(T)$.*

Proof. (i) The inclusions $\sigma_{\mathrm{wa}}(T) \subseteq \sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ hold for every $T \in L(X)$. Suppose that $\lambda \notin \sigma_{\mathrm{wa}}(T)$. Then $\lambda I - T \in \Phi_+(X)$ with $\mathrm{ind}\,(\lambda I - T) \leqslant 0$. The SVEP of T^* at λ entails that $q(\lambda I - T) < \infty$, see Remark 2.1, and by [1, Theorem 3.4] this implies that $\mathrm{ind}\,(\lambda I - T) \geqslant 0$. Therefore, $\mathrm{ind}\,(\lambda I - T) = 0$, i.e. $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$ and consequently, again by [1, Theorem 3.4], $p(\lambda I - T) < \infty$, thus $\lambda \notin \sigma_{\mathrm{b}}(T)$.

(ii) The inclusions $\sigma_{\mathrm{wa}}(T^*) = \sigma_{\mathrm{ws}}(T) \subseteq \sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ hold for every $T \in L(X)$. Suppose that $\lambda \notin \sigma_{\mathrm{wa}}(T^*)$. Then $\lambda I^* - T^* \in \Phi_+(X^*)$ with $\mathrm{ind}\,(\lambda I^* - T^*) \leqslant 0$. By duality $\lambda I - T \in \Phi_-(X)$ and by Remark 2.1 the SVEP of T at λ entails that $p(\lambda I - T) < \infty$. By [1, Theorem 3.4] we have $\mathrm{ind}\,(\lambda I - T) \leqslant 0$, thus $\mathrm{ind}\,(\lambda I^* - T^*) = -\mathrm{ind}\,(\lambda I - T) \geqslant 0$. Therefore, $\mathrm{ind}\,(\lambda I^* - T^*) = \mathrm{ind}(\lambda I - T) = 0$, and, again by [1, Theorem 3.4], we have $q(\lambda I - T) < \infty$, thus $\lambda \notin \sigma_{\mathrm{b}}(T)$. □

The following result has a crucial role in the sequel.

Theorem 2.7. *Suppose that $T \in L(X)$. Then the following statements hold*:

(i) *If T is polaroid and T^* has SVEP then property (w) holds for T.*
(ii) *If T is polaroid and T has SVEP then property (w) holds for T^*.*

Proof. (i) Note that by Corollary 2.45 of [1] we have $\sigma_{\mathrm{a}}(T) = \sigma(T)$. Suppose first that $\mathrm{iso}\,\sigma(T) = \emptyset$. Then $\pi_{00}(T) = \emptyset$. We show that also $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T)$ is empty. By Theorem 2.6 we have $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$ and the last set is empty, since $\sigma(T)$ has no isolated points. Therefore, T satisfies property (w).

Consider the other case, $\mathrm{iso}\,\sigma(T) \neq \emptyset$. Suppose that $\lambda \in \pi_{00}(T)$. Then λ is isolated in $\sigma(T)$ and hence, by the polaroid condition, λ is a pole of the resolvent of T, i.e. $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. By assumption $\alpha(\lambda I - T) < \infty$, so by [1, Theorem 3.1] $\beta(\lambda I - T) < \infty$, and hence $\lambda I - T \in \Phi(X)$. Therefore, by Theorem 2.6, $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \sigma(T) \setminus \sigma_{\mathrm{wa}}(T)$. Conversely, if $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$ then λ is an isolated point of $\sigma(T)$. Clearly, $0 < \alpha(\lambda I - T) < \infty$, so $\lambda \in \pi_{00}(T)$ and hence T satisfies property (w).

(ii) First note that since T has SVEP then $\sigma_{\mathrm{a}}(T^*) = \sigma_{\mathrm{s}}(T) = \sigma(T) = \sigma(T^*)$, see Corollary 2.45 of [1]. Suppose first that $\operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T^*) = \emptyset$. Then $\pi_{00}(T^*) = \emptyset$. By Theorem 2.6 we have $\sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{wa}}(T^*) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \emptyset$, so T^* satisfies property (w).

Suppose that $\operatorname{iso}\sigma(T) \neq \emptyset$ and let $\lambda \in \pi_{00}(T^*)$. Then λ is an isolated point of $\sigma(T^*) = \sigma(T)$, hence a pole of the resolvent of T^*, since T^* is polaroid by Theorem 2.5. By assumption $\alpha(\lambda I^* - T^*)^p < \infty$ and since the ascent and the descent of $\lambda I^* - T^*$ are both finite it then follows by [1, Theorem 3.1] that $\beta(\lambda I - T) = \alpha(\lambda I - T) < \infty$, so $\lambda I^* - T^*$ is Browder and hence also $\lambda I - T$ is Browder. Therefore, $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$ and by Theorem 2.6 it then follows that $\lambda \in \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{wa}}(T^*)$.

Conversely, if $\lambda \in \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{wa}}(T^*) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, then λ is an isolated point of the spectrum of $\sigma(T) = \sigma(T^*)$, $\lambda I - T \in B(X)$, or equivalently $\lambda I^* - T^* \in B(X^*)$. Since $\alpha(\lambda I^* - T^*) = \beta(\lambda I^* - T^*)$ we then have $\alpha(\lambda I^* - T^*) > 0$ (otherwise $\lambda \notin \sigma(T^*)$). Clearly, $\alpha(\lambda I^* - T^*) < \infty$, since by assumption $\lambda I^* - T^* \in W_+(X^*)$, so that $\lambda \in \pi_{00}(T^*)$. Thus T^* satisfies property (w). $\square$

The following example shows that in the statements (i) of Theorem 2.7 the assumption that T^* has SVEP cannot be replaced by the assumption that T has SVEP.

Example 2.8. Denote by R the unilateral right shift on $\ell^2(\mathbb{N})$ and define

$$U(x_1, x_2, \ldots) := (0, x_2, x_3, \ldots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Clearly, U is a quasi-nilpotent operator. Let $T := R \oplus U$. We have $\sigma(T) = \mathbf{D}$, $\mathbf{D}$ the closed unit disc of $\mathbb{C}$, so $\operatorname{iso}\sigma(T) = \pi_{00}(T) = \emptyset$ and hence T is polaroid. Moreover, $\sigma_{\mathrm{a}}(T) = \partial\sigma(T) \cup \{0\}$. By (4) T has SVEP at 0, as well as at the points $\lambda \notin \sigma_{\mathrm{a}}(T)$. Since T has SVEP at all points $\lambda \in \partial\sigma(T)$ it then follows that T has SVEP. Finally, $\sigma_{\mathrm{wa}}(T) = \partial\sigma(T)$ so $\sigma(T) \setminus \sigma_{\mathrm{wa}}(T) = \{0\} \neq \pi_{00}(T) = \emptyset$, thus T does not satisfy property (w).

Analogously, in the statements (ii) of Theorem 2.7 the assumption that T has SVEP cannot be replaced by the assumption that T^* has SVEP.

Example 2.9. Let us consider the left shift $L \in L(\ell^2(\mathbb{N}))$, and let U' be the adjoint of the quasi-nilpotent operator U defined in Example 2.8. We have $L' = R$, R the unilateral right shift. If we define $S := L \oplus U'$ then, as observed in Example 2.8 $S' = R \oplus U$ has SVEP (and hence the dual S^* has SVEP). From Example 2.8 we also know that $\sigma(S) = \overline{\sigma(S')} = \mathbf{D}$, so S is polaroid and S' has not property (w), or equivalently by Theorem 2.4, S^* has not property (w).

In the sequel by $H_0(T)$ we shall denote the *quasi-nilpotent part* of T, defined as

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

It is easily seen that $\ker(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$. Another important subspace in local spectral theory is given by *analytic core* of T, defined as the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and $\delta > 0$ for which $x = u_0$, and $Tu_{n+1} = u_n$ and $\|u_n\| \leqslant$

$\delta^n \|x\|$ for every $n \in \mathbb{N}$. It easily follows, from the definition, that $K(T)$ is a linear subspace of X and that $T(K(T)) = K(T)$. Note that, (see [4]),

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda, \tag{7}$$

and this implication is an equivalence whenever $\lambda I - T \in \Phi_{\pm}(X)$, see [4].

Theorem 2.10. *If $T \in L(X)$ is polaroid and $N \in L(X)$ is nilpotent such that $TN = NT$. Then $T + N$ is polaroid.*

Proof. Note first that the quasi-nilpotent part of every operator $S \in L(X)$ is invariant under nilpotent perturbations. In fact, let $x \in H_0(S)$ and suppose that $N^\nu = 0$ for some $\nu \in \mathbb{N}$. If $U := \sum_{k=0}^{\nu-1} c_{\nu,k} S^{\nu-1-k} N^k$, with suitable binomial coefficients $c_{\nu,k}$, then it is easily seen that $(S+N)^\nu = SU$, from which the following estimate follows

$$\|(S+N)^{\nu n} x\|^{\frac{1}{n}} \leqslant \|S^n x\|^{\frac{1}{n}} \|U^n x\|^{\frac{1}{n}}.$$

From this estimate we deduce that $\lim_{n\to+\infty} \|(S+N)^{\nu n} x\|^{\frac{1}{n}} = 0$, i.e. $x \in H_0(S+N)$, and hence $H_0(S) \subseteq H_0(S+N)$. The reverse inclusion follows by symmetry:

$$H_0(S+N) \subseteq H_0((S+N) - N) = H_0(S).$$

To show that $T + N$ is polaroid, let $\lambda \in \operatorname{iso} \sigma(T+N)$. It is well-known that the spectrum of an operator is invariant under nilpotent perturbations, so λ is an isolated point in $\sigma(T)$, hence a pole of the resolvent of T. Let $p := p(\lambda I - T) = q(\lambda I - T)$. By the first part of the proof and by Theorem 3.74 of [1] we have

$$H_0(\lambda I - T + N) = H_0(\lambda I - T) = \ker(\lambda I - T)^p.$$

Set $m := p\nu$. We show that

$$H_0(\lambda I - T + N) = \ker(\lambda I - T + N)^m. \tag{8}$$

The inclusion $\ker(\lambda I - T + N)^n \subseteq H_0(\lambda I - T + N)$ is true for all operators and for all $n \in \mathbb{N}$. Suppose that $x \in H_0(\lambda I - T + N) = \ker(\lambda I - T)^p$. Then we can write

$$(\lambda I - T + N)^m x = \sum_{i=m-p+1}^{m} [\mu_{i,m} (\lambda I - T)^{m-i} N^{i-\nu}] N^\nu x = 0,$$

with suitable binomial coefficients $\mu_{i,m}$, thus

$$H_0(\lambda I - T + N) = \ker(\lambda I - T)^p \subseteq \ker(\lambda I - T + N)^m$$

and hence (8) is proved.

We show now that $T + N$ is polaroid. Let $\lambda \in \operatorname{iso} \sigma(T+N)$. Then, by Theorem 3.74 of [1], we have

$$X = H_0(\lambda I - T + N) \oplus K(\lambda I - T + N) = \ker(\lambda I - T + N)^m \oplus K(\lambda I - T + N),$$

so that

$$(\lambda I - T + N)^m(X) = (\lambda I - T + N)^m (K(\lambda I - T + N)) = K(\lambda I - T + N).$$

Therefore, $X = \ker(\lambda I - T + N)^m \oplus (\lambda I - T + N)^m(X)$ and this decomposition implies, by Theorem 3.6 of [1], that $\lambda I - T + N$ has both finite ascent and descent, or equivalently that λ is a pole of the resolvent of $T + N$. $\square$

Theorem 2.11. *Suppose that $T \in L(X)$ is polaroid, $N \in L(X)$ a nilpotent operator commuting with T.*

(i) *If T has SVEP then $T^* + N^*$ satisfies property (w), or equivalently a-Weyl's theorem holds for $T^* + N^*$.*
(ii) *If T^* has SVEP then $T + N$ satisfies property (w), or equivalently a-Weyl's theorem holds for $T + N$.*

Proof. (i) If T has SVEP then $T + N$ has SVEP, see Corollary 2.12 of [1]. Moreover, by Lemma 2.10 $T + N$ is polaroid. By Theorem 2.7 it then follows that property (w) holds for $T^* + N^*$, or equivalently, since $T + N$ has SVEP, a-Weyl's theorem holds for $T^* + N^*$.

(ii) If T is polaroid then by Theorem 2.5 T^* is polaroid. Clearly, N^* is nilpotent, since $(N^*)^n = (N^n)^* = 0$ for some $n \in \mathbb{N}$. Therefore $T^* + N^*$ is polaroid, by Theorem 2.10. Since $T^* + N^*$ has SVEP, by Corollary 2.12 of [1], it then follows, by Theorem 2.7, that $T + N$ satisfies property (w), or equivalently a-Weyl's theorem. □

We can say much more than Theorem 2.11. Denote by $\mathscr{H}(\sigma(T))$ the set of all analytic functions defined on a neighborhood $\mathscr{U}$ of $\sigma(T)$, let $f(T)$ be defined by means of the classical functional calculus:

$$f(T) := \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda I - T)^{-1}\mathrm{d}\lambda,$$

where Γ is a contour in $\mathscr{U}$ that surrounds $\sigma(T)$.

Theorem 2.12. *Suppose that T is polaroid and $N \in L(X)$ a nilpotent operator commuting with T. If T^* has SVEP and $f \in H(\sigma(T))$ then property (w) holds for $f(T) + N$, or equivalently a-Weyl's theorem holds for $f(T) + N$.*

Proof. By Theorem 2.7, T satisfies property (w), or equivalently, by Theorem 2.2, T satisfies Weyl's theorem. The SVEP for T^* implies that $\sigma(T) = \sigma_{\mathrm{a}}(T)$, see [1, Corollary 2.45], so every isolated point of $\sigma_{\mathrm{a}}(T)$ is a pole of the resolvent of T. By [6, Theorem 2.22] then $f(T)$ satisfies property (w). Finally, by [3, Theorem 2.8] $f(T) + N$ satisfies property (w). Since $f(T^*) = f(T)^*$ has SVEP, see Theorem 2.40 of [1], by Theorem 2.2 it then follows that property (w) and a-Weyl's theorem for $f(T) + N$ are equivalent. □

Remark 2.13. It is somewhat meaningful to ask what we can say about the operators $f(T + N)$, always under the assumptions of Theorem 2.12. Now, if T is polaroid then $T + N$ is polaroid, by Theorem 2.10. Moreover, by $T^* + N^* = (T + N)^*$ has SVEP by Corollary 2.12 of [1]. Now, as it has been observed in the proof of Theorem 2.12, if $S := T + N$ is polaroid and $S^* = (T + N)^*$ has SVEP then S is a-polaroid. Hence by [6, Theorem 2.22] $f(T + N)$ satisfies property (w) for every $f \in \mathscr{H}(\sigma(T))$.

A bounded operator $T \in L(X)$ is said to have *property* $H(p)$ if for all $\lambda \in \mathbb{C}$ there exists a $p := p(\lambda) \in \mathbb{N}$ such that:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p.$$

This class of operators has been introduced in [18] and for the constant function $p(\lambda) := 1$ has been also studied in [7]. Clearly, from the implication (7), if T has property $H(p)$ then T has SVEP. Moreover T is polaroid, see [2, Lemma 3.3].

It should be noted that property $H(p)$ holds for a relevant number of Hilbert space operators. In [18, Example 3] Oudghiri observed that every *generalized scalar operator* and every *subscalar operator* T (i.e. T is similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces, we refer to [15] for definitions and results) has property $H(p)$, see [15] for definitions and properties. Consequently, property $H(p)$ is satisfied by p-hyponormal operators and log-hyponormal operators [16, Corollary 2], w-hyponormal operators [17], M-hyponormal operators [15, Proposition 2.4.9], and totally paranormal operators [7]. Also totally *-paranormal operators have property $H(1)$ [13]. Property $H(p)$ for $p := p(\lambda) = 1$ also holds for every multiplier $T \in M(A)$ of a commutative semi-simple Banach algebra A, and in particular for every convolution operator on $L^1(G)$, where G is a compact Abelian group [7].

A bounded operator $T \in L(X)$ on a Banach space X is said to be *paranormal* if

$$\|Tx\|^2 \leqslant \|T^2x\|\|x\| \quad \text{holds for all } x \in X.$$

The class of paranormal operators properly contains a relevant number of Hilbert space operators, among them p-hyponormal operators, log-hyponormal operators, and the class of operators satisfying the condition $|T|^2 \leqslant |T^2|$, see [12] and [10]. Note that, in general, paranormal operators do not satisfy property $H(p)$, see [5] for a counter-example. A bounded operator $T \in L(X)$ is said to be *algebraically paranormal* if there exists a non-trivial polynomial h such that $h(T)$ is paranormal.

Every paranormal operator on a Hilbert space has SVEP. This may be easily seen as follows: if $\lambda \neq 0$ and $\lambda \neq \mu$ then, by Theorem 2.6 of [8], we have

$$\|x + y\| \geqslant \|y\| \quad \text{for all } x \in \ker(\mu I - T) \text{ and } y \in \ker(\lambda I - T). \tag{9}$$

It then follows that if U is an open disc and $f : U \to X$ is an analytic function such that $0 \neq f(z) \in \ker(zI - T)$ for all $z \in U$, then f fails to be continuous at every $0 \neq \lambda \in U$. By Theorem 2.40 of [1] it then follows that every algebraically paranormal satisfies SVEP.

Recall that a bounded operator T is said to be *algebraic* if there exists a non-trivial polynomial h such that $h(T) = 0$. From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators K are algebraic; more generally, if K^n is a finite rank operator for some $n \in \mathbb{N}$ then K is algebraic. Clearly, if T is algebraic then its dual T^* is algebraic, as well as T' in the case of Hilbert space operators.

Theorem 2.14. *Suppose that $T \in L(X)$ and $K \in L(X)$ is an algebraic operator commuting with T.*

(i) *If T has SVEP then $T + K$ has SVEP.*
(ii) *If T is algebraically paranormal, or $T \in H(p)$, then $T + K$ is polaroid.*

Proof. (i) Let $\sigma(K) = \{\mu_1, \ldots, \mu_n\}$. Denote by P_i the spectral projection associated with K and with the spectral set $\{\mu_i\}$. If $Y_i := P_i(H)$ and $Z_i := \ker P_i$, then $H = Y_i \oplus Z_i$, the closed subspaces Y_i and Z_i are invariant under T and K, and $\sigma(K|Y_i) = \{\mu_i\}$. Define $K_i := K|Y_i$ and $T_i := T|Y_i$. Clearly, the restrictions T_i and K_i commute for every $i = 1, 2, \ldots, n$. Let h be a polynomial such that $h(K) = 0$. Then $h(K_i) = h(K)|Y_i = 0$, and the equalities

$$\{0\} = \sigma(h(K_i) = h(\sigma(K_i)) = h(\{\mu_i\})$$

entail that $h(\mu_i) = 0$. Write

$$h(\mu) = (\mu_i - \mu)^\nu q(\mu) \quad \text{with } q(\mu_i) \neq 0.$$

Then $0 = h(K_i) = (\mu_i I - K_i)^\nu q(K_i)$ where the operators $q(K_i)$ is invertible. Therefore $(\mu_i I - K_i)^\nu = 0$, hence the operators $N_i := \mu_i I - K_i$ are nilpotent for all $i = 1, 2, \ldots, n$. Note that

$$T_i + K_i = (\mu_i I + T_i) + (K_i - \mu_i I) = \mu_i I + T_i - N_i. \tag{10}$$

Since SVEP is inherited by restrictions to closed invariant subspaces, T_i has SVEP and hence, by [1, Corollary 2.12] also $T_i + K_i = \mu_i I + T_i - N_i$ has SVEP for all $i = 1, 2, \ldots, n$. By Theorem 2.9 of [1] it then follows that

$$T + K = \bigoplus_{i=1}^{n} T_i + K_i$$

has SVEP.

(ii) Suppose that T is algebraically paranormal. Let $\lambda \in \operatorname{iso}\sigma(T + K)$. Since $\sigma(T + K) = \bigcup_{i=1}^{n} \sigma(T_i + K_i)$ then $\lambda \in \operatorname{iso}\sigma(T_j + K_j)$ for some positive integer $1 \leqslant j \leqslant n$ and hence $\lambda - \mu_j \in \operatorname{iso}\sigma(T_j + K_j - \mu_j I)$. The restriction to a closed invariant subspace of an algebraically paranormal is algebraically paranormal, so T_j is polaroid. Since, as observed before, $\mu_j I - K_j$ is nilpotent then, by Theorem 2.10, also $T_j + K_j - \mu_j I$ is polaroid. Therefore $\lambda - \mu_j$ is a pole of the resolvent of $T_j + K_j - \mu_j I$, so there exists by Theorem 3.74 of [1] a positive integer p_j such that

$$H_0[(\lambda - \mu_j)I - (T_j + K_j - \mu_j I)] = H_0(\lambda I - (T_j + K_j)) = \ker(\lambda I - (T_j + K_j))^{p_j}.$$

Therefore, taking in to account that $H_0(\lambda I - (T_i + K_i) = \{0\}$ if $\lambda \notin \sigma(T_i + K_i)$, we have

$$\begin{aligned} H_0(\lambda I - (T + K)) &= \bigoplus_{i=1}^{n} H_0(\lambda I - (T_i + K_i)) = \bigoplus_{i=1}^{n} \ker(\lambda I - (T_i + K_i))^{p_i} \\ &= \ker(\lambda I - (T + K))^p, \end{aligned}$$

where $p := \max\{p_1, p_2, \ldots, p_n\}$. Arguing as in the proof of Theorem 2.10 it then follows that λ is a pole of the resolvent of $T + K$. The assertion concerning $H(p)$-operators may be proved in a similar way, just observe that if $T \in H(p)$ then every restriction of T to a closed invariant subspaces has property $H(p)$ and hence is polaroid, see [18]. $\square$

In [5] it has been proved that if $T \in L(H)$ is paranormal, K is an algebraic operator commuting with T, then Weyl's theorem holds for $T + K$. Something more can be said.

Theorem 2.15. *Suppose that $T \in L(H)$, H a Hilbert space and $K \in L(H)$ is an algebraic operator commuting with T.*

(i) *If T is algebraically paranormal then property (w) holds for $T' + K'$.*
(ii) *If T' is algebraically paranormal then property (w) holds for $T + K$.*

Proof. (i) If T is algebraically paranormal then T has SVEP and hence $T+K$ has SVEP. Moreover, T is polaroid so also $T+K$ is polaroid. By Theorem 2.7, part (i), then property (w) holds for T^*+K^* and this is equivalent, by Lemma 2.4, to saying that property (w) holds for $T'+K'$.

(ii) If T' is algebraically paranormal then T' has SVEP and hence $T'+K'$ has SVEP, equivalently T^*+K^* has SVEP. Moreover, $T'+K'$ is polaroid, so, by Theorem 2.5, $T+K$ is polaroid. By Theorem 2.7 it then follows that property (w) holds for $T+K$. □

In [19] it has been proved that if $T \in H(p)$, K is an algebraic operator commuting with T, then Weyl's theorem holds for $T+K$. A similar result of that of Theorem 2.15 holds for operators $T \in H(p)$.

Theorem 2.16. *Suppose that $T \in L(X))$ and $K \in L(X)$ is an algebraic operator commuting with T.*

(i) *If $T \in H(p)$ then property (w) holds for T^*+K^*.*
(ii) *If $T^* \in H(p)$ then property (w) holds for $T+K$.*

Proof. Proceed as in the proof of Theorem 2.15. □

Theorem 2.15 and Theorem 2.16 may be somewhat extended as follows: Recall that a subspace M of X is said to be *orthogonal* (in the sense of Birkhoff) to an other subspace N of X, $M \perp N$ if $\|x\| \leqslant \|x+y\|$ for all $x \in M$, $y \in N$. This asymmetric definition coincides with the usual concept of orthogonality in the case that X is a Hilbert space. M and N are said to be *mutually orthogonal*, $M \perp_m N$ if $M \perp N$ and $N \perp M$. From (9) it then follows that paranormal operators satisfy the *orthogonality condition*

$$\ker(\mu I - T) \perp_m \ker(\lambda I - T) \quad \text{forall } \lambda \neq \mu, \quad \lambda \neq 0, \tag{11}$$

and the same argument used for paranormal operators shows that this condition entails SVEP for T. Recall that $T \in L(X)$ is said to be *normaloid* if $\|T\| = r(T)$, $r(T)$ the spectral radius of T. A bounded operator T is said *completely hereditarily normaloid*, $T \in \mathscr{C}HN$, if either every part, and (also) every invertible part, of T is normaloid, or for all $\lambda \in \mathbb{C}$ every part of $\lambda I - T$ is normaloid.

Theorem 2.17. *Suppose that $T \in \mathscr{C}HN$ satisfies the orthogonality condition* (11). *If $K \in L(X)$ is algebraic and commutes with T then T^*+K^* satisfies property (w).*

Proof. Every $T \in \mathscr{C}HN$ is polaroid, so $T+K$ is polaroid. The orthogonal condition (11) entails that T has SVEP, hence also $T+K$ has SVEP, by Lemma 2.14. The result then follows from Theorem 2.7, part (ii). □

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